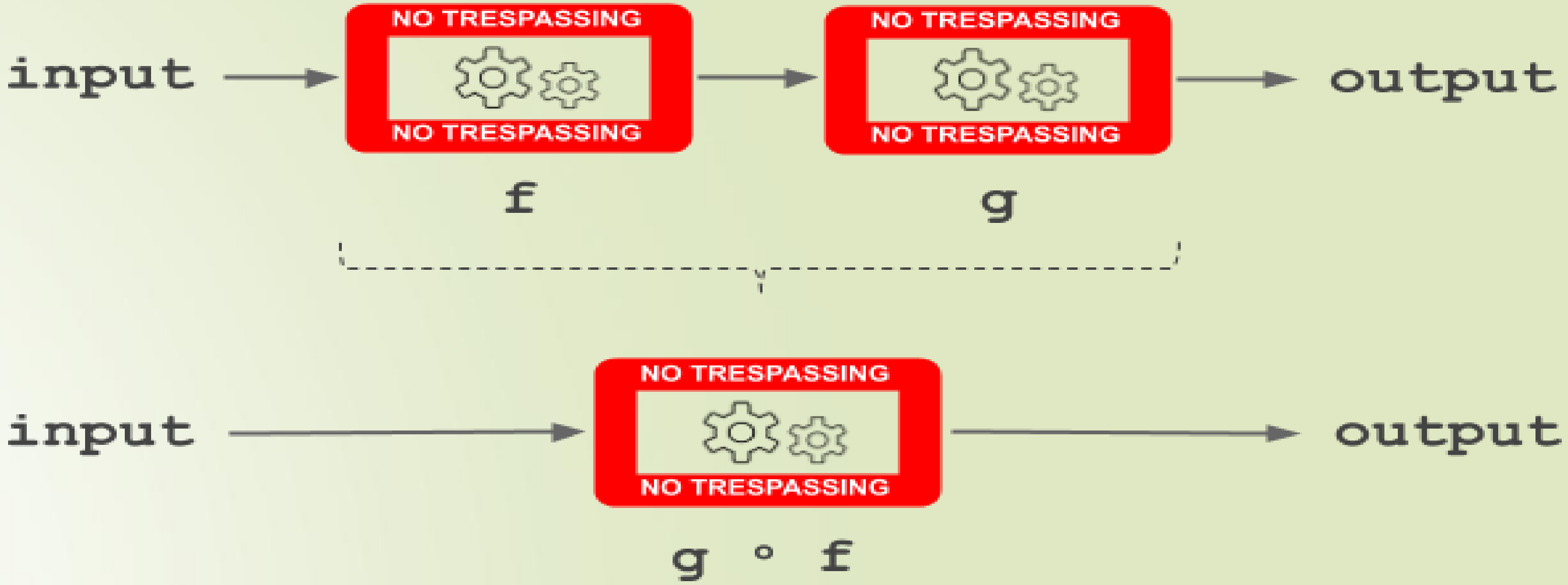
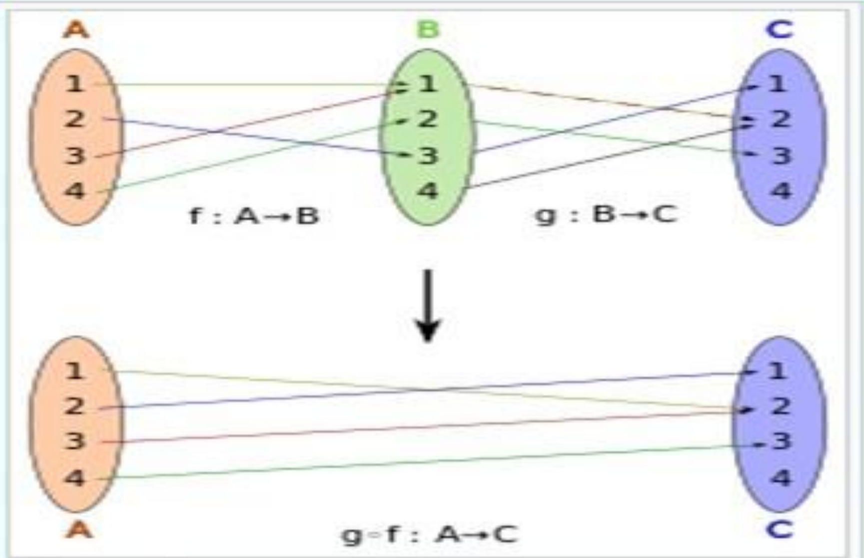
COMPOSITION OF FUNCTIONS AND INVERTIBLE FUNCTIONS



DEFINITION OF COMPOSITION OF FUNCTIONS

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be two functions. Then the composition of f and g, denoted by gof, is defined as the function $gof: A \longrightarrow C$ given by

$$gof(x) = g(f(x)),\ \forall\ x \in A.$$



Concrete example for the composition of two functions.

Ex 1.3, 3

Find gof and fog, if

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

$f(x) = 8x^3 \quad g(x) = x^{\frac{1}{3}}$

$f(x) = 8x^3$

$f(g(x)) = 8\, g(x)^3$

$\boldsymbol{fog(x)} = 8(x^{\frac{1}{3}})^3$

$= 8\, x^{\frac{1}{3} \times 3}$

$= \mathbf{8x}$

$g(x) = x^{\frac{1}{3}}$

$g(f(x)) = f(x)^{\frac{1}{3}}$

$\boldsymbol{gof\,(x)} = (8x^3)^{\frac{1}{3}}$

$= ((2x)^3)^{\frac{1}{3}}$

$= (2x)^{3 \times \frac{1}{3}}$

$= \mathbf{2x}$

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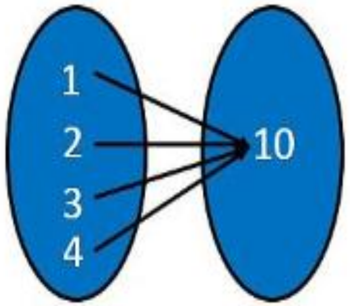
State with reason whether following functions have inverse

(i) f: {1, 2, 3, 4} → {10} with f = {(1, 10), (2, 10), (3, 10), (4, 10)}

A function has inverse if it is one-one and onto

Check one one

f = {(1, 10), (2, 10), (3, 10), (4, 10)}



Since all elements have image 10,

They do not have unique image

∴ f is **not one-one.**

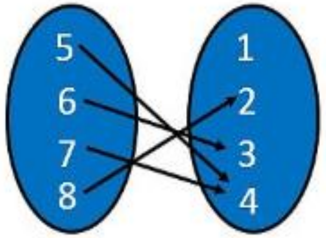
Since, f is not one-one, it **does not have an inverse.**

(ii) g: {5, 6, 7, 8} → {1, 2, 3, 4} with g = {(5, 4), (6, 3), (7, 4), (8, 2)}

A function has inverse if it is one-one and onto

Check one one

g = {(5, 4), (6, 3), (7, 4), (8, 2)}



REASON ????

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Ex 1.3, 5

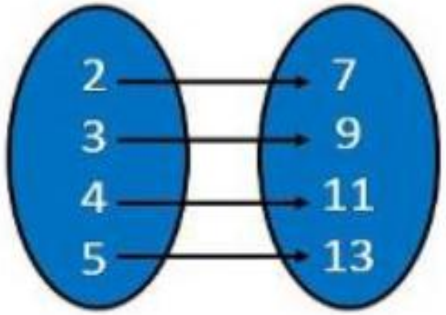
State with reason whether following functions have inverse

(iii) h: {2, 3, 4, 5} → {7, 9, 11, 13} with h = {(2, 7), (3, 9), (4, 11), (5, 13)}

A function has inverse if it is one-one and onto

Check one one

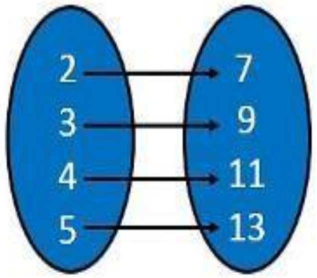
h = {(2, 7), (3, 9), (4, 11), (5, 13)}



Since each element has unique image,

h is **one-one**

Check onto



Since for every image, there is a corresponding element,

∴ h is **onto**.

Since function is both one-one and onto

it will have inverse

h = {(2, 7), (3, 9), (4, 11), (5, 13)}

h^{-1} = {(7, 2), (9, 3), (11, 4), (13, 5)}

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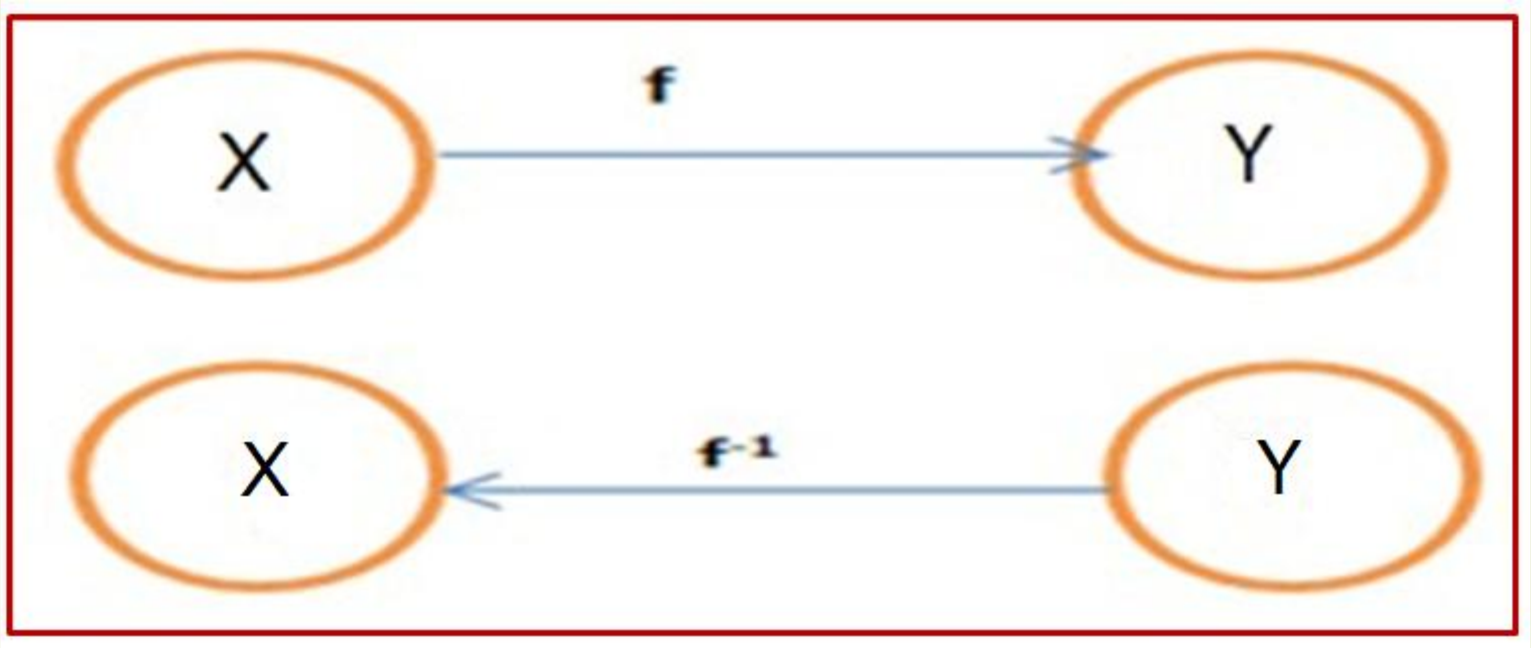
DEFINITION OF AN INVERTIBLE FUNCTION

- A function $f: X \longrightarrow Y$ is defined to be *invertible*, if there exists a function $g: Y \longrightarrow X$ such that $gof = I_X$ and $fog = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}.



Ex 1.3, 6

Show that f: [−1, 1] → **R**, given by f(x) = $\frac{x}{x+2}$ is one-one. Find the inverse of the function f: [−1, 1] → Range f.

Let *y* be an arbitrary element of range *f*.

$y = f(x)$ for same $x \in [-1, 1]$

$\Rightarrow y = \frac{x}{x+2}$

$\Rightarrow xy + 2y = x$

$\Rightarrow x(1-y) = 2y$

$\Rightarrow x = \frac{2y}{1-y}, y \neq 1$

Now, let us define *g*: Range *f* → [−1, 1] as

$g(y) = \frac{2y}{1-y}, y \neq 1.$

Now, $(gof)(x) = g(f(x)) = g\left(\frac{x}{x+2}\right) = \frac{2\left(\frac{x}{x+2}\right)}{1-\frac{x}{x+2}} = \frac{2x}{x+2-x} = \frac{2x}{2} = x$

$(fog)(y) = f(g(y)) = f\left(\frac{2y}{1-y}\right) = \frac{\frac{2y}{1-y}}{\frac{2y}{1-y}+2} = \frac{2y}{2y+2-2y} = \frac{2y}{2} = y$

$\therefore gof = I_{[-1,1]}$ and $fog = I_{\text{Range} f}$

$\therefore f^{-1} = g$

$\Rightarrow f^{-1}(y) = \frac{2y}{1-y}, y \neq 1$

$\because g(y) = \frac{2y}{1-y} \Rightarrow g(f(x)) = \frac{2f(x)}{1-f(x)}$

$\because f(x) = \frac{x}{x+2} \Rightarrow f(g(y)) = \frac{g(y)}{g(y)+2}$

$gof = x \longrightarrow I_X$

$fog = y \longrightarrow I_Y$

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Ex 1.3, 9

Consider f: $R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with the inverse f^{-1} of given f by $f^{-1}(y) = \frac{(\sqrt{y+6}) - 1}{3}$.

$f: R_+ \to [-5, \infty)$ is given as $f(x) = 9x^2 + 6x - 5$.

Let y be an arbitrary element of $[-5, \infty)$.

Let $y = 9x^2 + 6x - 5$

$\Rightarrow y = (3x+1)^2 - 1 - 5$

$= (3x+1)^2 - 6$

$\Rightarrow y + 6 = (3x+1)^2$

$\Rightarrow 3x + 1 = \sqrt{y+6}$ [as $y \geq -5 \Longrightarrow y + 6 > 0$]

$\Rightarrow x = \frac{(\sqrt{y+6}) - 1}{3}$.

range $f = [-5, \infty)$.

Let us define $g: [-5, \infty) \to R_+$ as $g(y) = \frac{(\sqrt{y+6}) - 1}{3}$

Now, $(gof)(x) = g(f(x))$

$= g(9x^2 + 6x - 5)$

$= g((3x+1)^2 - 6)$

$= \sqrt{(3x+1)^2 - 6 + 6} - 1$

$= \frac{3x + 1 - 1}{3} = \frac{3x}{3} = x$

Thus $(gof)(x) = x$

and $(fog)(y) = f(g(y))$

$= f\left(\frac{\sqrt{y+6} - 1}{3}\right)$

$= \left[3\left(\frac{\sqrt{y+6} - 1}{3}\right) + 1\right]^2 - 6$

$= \left(\sqrt{y+6}\right)^2 - 6$

$= y + 6 - 6$

$= y$

Thus $(fog)(y) = y$

$\therefore gof = x = I_R$ and $fog = y = I_{Range\ f}$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \left(\frac{(\sqrt{y+6}) - 1}{3}\right)$$

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HOME ASSIGNMENT

- EXERCISE – 1

Q.3 NO. 7, 8, 11, 14

EXAMPLE – 25

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